Developing the angle concept through investigations

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Children in Year 2 and 4 were developing the concept of angle through their responsiveness, manipulating of materials, and interactions with others. Their selective attention to aspects of the materials, interactions, or their own imagery and thinking assisted the development of the concept.

Introduction

Piaget and Inhelder (1956) concluded from their studies that there is no doubt

> that it is the analysis of the angle which marks the transition from topological relationships to the perception of Euclidean ones. It is not the straight line itself which the child contrasts with round shapes, but rather that conjunction of straight lines which go to form an angle. (p. 30)

Students intuitively recognise an angle at an early age but need to develop the concept, associated language, imagery, angle contexts, and related constructs such as size of angle.

Children (e.g. Years 3 and 6) have difficulty in recognising equal angles in differing orientations, with differing arm lengths, and when embedded in figures (Outhred, 1987). Outhred found that right angles that were not aligned with the horizontal axis were often not recognised. Similar difficulties have been mentioned in reports of other studies (Fuys, Geddes, & Tischler, 1988; Mitchelmore, 1989; Pegg & Davey, 1989). More pre-measurement informal experiences with angles, their size, orientation, and contexts were recommended by Close (1982). The pre-measurement activities used in the current study encourage perception of angles in figures.

The ideas of *being pointy* and of *being a corner* or right angle is understood relatively early (Davey & Pegg, 1991). The right angle is not regarded as an angle by some students, others believe that only right angles are angles, and others think that only acute angles are angles. Teaching activities need to help children see right angles as special angles (Mitchelmore, 1992). The present study placed right angles in the family of angles and encouraged informal comparison of angles.

Year 2 and 4 children have a good global understanding of different angle contexts, and they improve significantly in their recognition of angle-related similarities (Mitchelmore,1993, 1994). Children most easily recognise similarities between situations where both lines are physically present (e.g., crossing) and are nearly as able when one line is present and one has to be imagined (e.g. a slope), but they find it difficult to relate turning, for example, of the body to face from one direction to another, (both lines to be imagined) to other angle contexts. Mitchelmore and White (1995) explained the development of general angle concepts in terms of abstraction. The angle concept is thought to develop gradually as children recognise more and deeper similarities between physical angle experiences, going through three successive stages: classification into physical angle situations such as walking up a slope and using scissors; then into separate angle contexts described as *sloping* and *crossing*; and finally into a general angle concept which includes all contexts. Turning, according to Mitchelmore, was more likely to be interpreted as a static anglerather than direction turning. In the present study, turning was related to static angles of shapes and other angle contexts were not used.

Abstractions are generally associated with concept imagery which needs to be dynamic and multifaceted if it is to encapsulate the concept adequately (Battista & Clements, 1991; Owens, 1994). The importance of mental imagery in estimation of angle size was illustrated by sixth and eighth grade students who imagined a protractor, a right angle, a half turn, or an angle of a polygon (Mansfield & Happs, 1992).

For two questions on angles, one on recognising a straight angle and the other on the angle of slope to the horizontal, only 74% and 64% respectively of Year 6 students New South Wales were successful (Owens, 1997). These data suggest that it is apparent that students have difficulties recognising angles in everyday situations, although it is recognised that the results could indicate that students have difficulties understanding the words: *straight angle* (*straight up or straight angle* is sometimes used for horizontally oriented right angles) and *horizontal*. Language difficulties were apparent when sixth grade students gave descriptions using non-standard vocabulary during Fuys et al.'s (1988) study although more formal language had been used in class.

Language difficulties make it hard to study the development of the concept of angle. Some researchers (e.g., Fuys et al., 1988; Pegg & Davey, 1989; van Hiele, 1986) have claimed that the type of language and the concept images which students use relate to a particular level(e.g. Van Hiele, 1986, levels). Language, however, can act as a mediator to differentiate a particular stimulus and promote attention. The connections between language and image may not be tied by level but, indeed, images and language could be regarded as limited conceptions promoting attention. The study to be described in this paper illustrated the importance of language in concept development and in focusing attention during problem solving and learning. There was a language difficulty also in using the word *angle* with young children because it seemed to have no meaning from earlier experiences. In an attempt to overcome this problem the word *point* was also used to refer to angle. However, this sometimes limited children's thinking about angles.

Clements and Battista (1992) have argued that students' learning can be explained in terms of cognitive processing rather than in terms of levels of development. It is, therefore, important that a qualitative study develop an understanding of students' conceptual learning by allowing the theoretical perspective to emerge from the data of the study.

The Study

The study focused on how students learnt from spatial problem-solving activities. The activities were designed to promote spatial thinking, especially imagery, problem solving, and analysis. Several concepts were developed during the study and this report will focus on the development of the concept of angles.

The qualitative study involved students in Years 2 and 4 in Australia and Papua New Guinea. The activities given to the students during 11 sessions were open-ended spatial problem-solving activities which improved students two-dimensional spatial thinking (Owens, 1993a). The type of problem-solving experiences encounted by the students can be illustrated by those posed for the well-known seven-piece tangram set made of three-sizes of isosceles right-angled triangles, a square, and a parallelogram (see Figure 1). Students were to look for similarities and differences between the pieces, to make the larger pieces from the smaller pieces, to make different-sized squares, triangles, and rectangles. Later they were asked to order the angles in size (small, middle-sized, and large) and make the shapes with sticks and matchsticks. Similar activities were completed using pattern-block sets containing squares, equilateral triangles, isosceles trapezia and two types of rhombi.

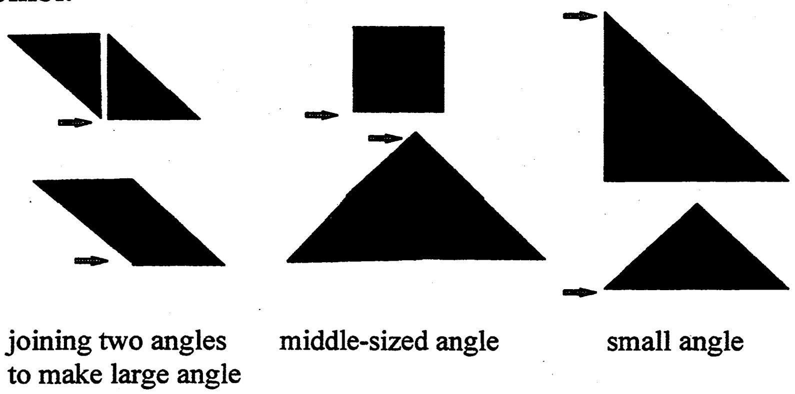

Figure 1. The angles of the seven-piece tangram set.

Students were assisted in recognising angles by turning their first finger, away from their thumb to mark the angles on the pieces as illustrated in Figure 2.

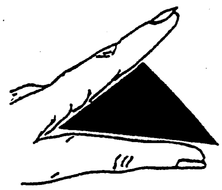

Figure 2. Using thumb and finger to note an angle on a tangram piece.

Several groups of students were purposefully selected as the qualitative aspects of learning unfolded. Several categories of thinking such as imagining and affect were defined as important from the initial sample of students who worked by themselves and provided immediate retrospective recall and spontaneous comments (Owens, 1991).

In the next stage, the effects of interaction were considered. Two groups of three children in Year 2 and two in Year 4 were given the problems and stimulated to recall how they were thinking immediately after solving each problem. One group from each year worked as a cooperative small group and the other as individuals but able to talk to their peers. Classroom contexts were considered when similar groups were used in classes in three schools from one region of Sydney, in a school from another suburban region, and in another cultural setting, namely in Papua New Guinea.

Students were video-taped while engaged in the activities and these tapes were analysed. Each incident during which there was a small development in problem solving was categorised. The categories included interaction with materials, student-student and student-teacher interaction, concepts, imagery, heuristics, and affect. The subcategories were developed during the grounded-theory research and linked to the literature review.

Results and Discussion

Examples have been chosen to illustrate how a concept, in this case that of an angle, can be developed in terms that are not associated with levels of development but with cognitive processing, and influences of the context of learning.

For Dora, in Year 2, there was a conflict between her perception of the materials and what she had observed a fellow student doing. Dora set out to compare the angles of the tangram pieces. The following notes described her participation in the activity.

1.01 Dora tells her teacher that her fingers have to be spread further apart for the large angle of the parallelogram. She gathers together most of the right angles. She puts her thumb and finger around them.

1.02 She seems a little exasperated for she is unsure about what is expected of them. She discusses with her friend about recording the angles—small, middle, and large—in her book and she draws the two arms of the small angle.

1.03 She knows that, beside the small triangle, both the big triangle and the parallelogram have a small angle.

1.04 She compares the right angle with the small angle but her friend calls the right angle large, so the teacher reassures her that the right angle is in the middle....

1.05 When the teacher asks her for the large angle, she picks up the parallelogram and claims she had it before her friend....

1.06 She joins up angles to make a right angle, and is pleased about doing this.

1.07 Then she joins the two triangles to make a parallelogram ... and shows this to the teacher as the large angle. (See Figure 1 configurations)

It is clear that this Year 2 child had perceived the differences between different angles on the various shapes. She was able to relate size of the angle to the position of the arms of the angle. The concrete materials helped her to see the same angle in different shapes (paragraphs 1.01, 1.03, and 1.04), and she was able to mark the angles with her fingers and to draw them (paragraph 1.02). She also compared angles with other angles represented by the pieces and drawings (paragraph 1.04). The above account also draws attention to a common affective response to making shapes or angles, that of pleasure and excitement following a measure of success with an activity (paragraphs 1.06 and 1.07). It is as if the concrete materials were often being used by students as a means of confirming and celebrating their abstract idea of an angle.

The materials provided physical representations of angles of different magnitudes, yet they were not sufficient to enable the children to appreciate the teacher's words about size of angles. In other words, the physical representation was not enough, even when combined with the teacher's words, for children to understand the meaning of different sizes of angles. First the children had to focus on or disembed the angle from the rest of the shape and then they had to construct their own meanings. Suggestions made by the teacher encouraged students not only to observe but also to check equivalence and to compare angles by overlaying them. For example, Jodie and James in Year 2, on being given the tangram set, immediately checked the points of the pieces in the same way as the teacher had done during the pretest. (Victor the third member of their group was away sick during this first session.)

In the session which will now be described, both Victor and James had to test their ideas of the size of an angle before they could clarify their understandings. The confusion lay in the use of the words *large, middle-sized,* and *small* to refer to the size of the pieces, the size of the side lengths, and the size of angles on the pieces. Victor was also completing the task set the other children in his absence, namely making the larger triangle with the other pieces. (The other children had apparently described the task to him).

2.01 Victor has gathered together some pieces, as if matching all the small angles.

2.02 James has picked up the small triangle for the small angle and points to its right angle. . . . The teacher asks for the middle-sized angle, he picks up the middle-sized triangle.

2.03 Victor picks up the parallelogram for the middle-sized angle, discards it, and then chooses the middle-sized triangle for the middle-sized angle. He puts his finger and thumb around the right angle and says "This is the middle angle." James watches.

2.04 Jodie is quite clear about which angle is smallest and which is middle-sized and she has been drawing both the small and middle-sized angles in the book.

2.05 James matches several angles against the drawing of the small angle.

2.06 Meanwhile Victor picks up the parallelogram and runs his finger along the long side. "This is the biggest" but he places it on the large triangle together with the two small triangles to cover the large triangle. He picks up the large triangle as if showing the small angle, "This is the biggest."

2.07 Jodie takes it and says "No, it isn't," and places it on the drawing of the small angle. The teacher confirms, "That's the small angle."

2.08 Meanwhile Victor has picked up the parallelogram, tests it against the drawings of the small and middle-sized angles and draws the large angle into the book.

2.09 He goes back and draws along the whole length of the arms.

2.10 The teacher suggests they make points (angles) by joining smaller points together. Victor puts points on top of each other.

2.11 James puts two small points together. The teacher says he has made the middle-sized one, praises him, and suggests that he can draw it in their book. James nods his head but he doesn't look convinced that he knows what he did. He puts two more together on top of the middle-sized point of the large triangle.

2.12 Meanwhile Victor has been trying to cover the large triangle, and he can see how to make it with the square and two triangles.

2.13 Jodie has now made the big angle with the angles of the two small triangles. (Figure 1.)

The discussion indicated how Victor, who seemed to know what was meant by the size of points (the word generally used by these children to refer to *angle*), temporarily considered that he should be comparing the size of the sides of the shapes (paragraphs 2.06 and 2.09). The interaction between students helped Victor to clarify what was meant by "the point of the same size" (paragraph 2.07). James, who had been able to match points in the first activity, began this later session by choosing the wrong points, largely because he was choosing the small or middle-sized triangles (paragraph 2.02). He soon established the meaning by listening to the teacher and to Jodie (paragraph 2.04) and by checking points with the drawing (paragraph 2.05). Later, although he successfully joined points and the teacher talked about joining angles to make new ones, he was only sure that he had the correct idea when the teacher encouraged him to show his new angle and praised him for his work (paragraph 2.10). Through manipulations, Victor (paragraph 2.12) spent some time relating the operational concept of making angles by joining together two angles to the making of congruent shapes.

Later the group made shape outlines and Victor explained that James had not made a right-angled triangle as James had thought but that he had just made an equilateral triangle in another orientation. Victor himself had made the right-angled isosceles triangle with the long side horizontal and he checked it with the tangram piece which he put on top ("a lid," he called it). The teacher asked the children what was meant by bigger. Jodie replied more spread out and picked up the tangram right-angled triangle and the pattern-block equilateral triangle, put one on top of the other and said, "See it is bigger."

The expectation of the students was to decide on the different sizes of the angles and they focused their attention on this responding by manipulating and discussing. With the later experiences there were still clarifications to be made especially when Victor was intent on completing the previous activity for which he was absent. When asked about his answers to the questions on angles in the posttest (the shapes now represented on paper), he had not initially noted the right-angle on a triangle in the same turned position which he had made with matchsticks and sticks. Later, when the page was turned for him, he said he could now see it was the same shape so it had to be the same angle. For another question, he noted that two right-angles were the same, "It is blunt like this one." His concepts of equality of angles involved the congruency of shapes, an informal description of the angles, visual imagery of the angle in different orientations, representations of angles by opening fingers and using sticks, an angle as the joining of smaller ones, and use of the word, for example, *large* to refer to different aspects of a triangle (an angle, overall size, length of side).

Students frequently noticed the equality of angles more that that of sides. The tendency to notice angles but not sides was due to holistic imagery and to an inability to disembed sides from the rest of the shape (Owens, 1993b; 1994; Owens & Clements, in press). The following is an extract from Year 2 children working next to each other with their own tangram sets.

3.01 Jonah makes a large parallelogram from the two large triangles. Sam says it is like the parallelogram piece which he picks up.

3.02 Lois makes her own parallelogram at a distance and watches as Sam matches the various angles of the parallelograms; she does likewise.

Informally Sam was showing similarity meant same size angles; he was recognising and making use of his understanding of size of angle. Students generally recognised angles on pieces which were likely to fit an angle to complete a jigsaw shape.

Although angles were perceptually strong, they were often difficult to describe. Students often said they were corners (right angles) or sharp. "The sharper it is the bigger it is" thought some, but Jodie, in fact, deliberately took on the opposite word to give the correct sense of size and stated that "the flatter it is the bigger it is,". Students commented on the spread of arms of an angle, represented by thumb and fore-finger, to explain why two angles were not equal. The teacher's interaction with the Year 2 students sparked a high degree of understanding of the size of angles and overall shapes. In the groups of three, who were asked to explain their thinking for every session, all the children (including the Year 2 children and the lower ability children) grasped the comparison and size of angles, and made other angles. In the classroom situation, fewer children were really sure about what they were doing, suggesting that not only did they have difficulties in disembedding the angles from the shape but also that student-teacher interactions assisted learning of the angle concept. This interaction was one of the influences on students' cognitive processing, especially on their selective attention. Words also gave students a means of marking aspects of their manipulations and what they were noticing.

Aspects of Problem-Solving

Responsiveness

Each of the above extracts illustrate the importance of students' responsiveness during active engagement in problem-solving activities, precipitated by their own thinking and feeling. It is a mental action for a specific context; a part of an interaction. It is manifest in physical responses such as expressions, movements, and words which, in turn, influence the context. Responsiveness is the movement forward, the risk-taking of problem-solving. Often multiple thoughts have to be held for consideration and action over several seconds or minutes until the context reacts to the development. The context change might be a verbal reply from a friend or the new position of materials when acted upon. Responsiveness implies a degree of understanding of the situation as well as involvement and interest in the activity (Owens, 1994).

Responsiveness results from a combination of cognitive processes which include attending, perceiving, listening, looking, visual imagining, conceptualising, intuitive thinking, and heuristic processing (such as establishing the meaning of the problem, developing tactics, self-monitoring and checking). Cognitive processing also incorporates affective processes such as reactions to the organisation of the classroom and to success, confidence, interest, and tolerance of open-ended situations. Problem-solving episodes or points involving critical change in thinking are likely to involve both changes in affect and changes in understanding.

Selective Attention

One particular aspect of cognitive processing, namely selective attention, gradually emerged from the study as extremely important in problem solving. In each of the above episodes, the students attended to particular aspects that affected their understanding of an angle. A student's attention may have been focused, for example, on the angle, the whole piece, its shape, the sides, a comment of a friend or the teacher, the result of an action, or their own thoughts and imagery. Turning fingers apart was used by students to focus attention on the size of static angles of shapes because turning is operational and hence a means by which children kinaesthetically appreciated and visualised the concept.

Selective attention initially links long-term memory with sensed perceptions, and then relates the perception to what is stored in memory. Selective attention guides a student's responsiveness to a problem and leads students to choose between acts. Actions are performed one at a time although several features can be perceived at the same time (Kaufmann, 1979). In the study, imagery was an important aspect of problem solving because it often (a) guided students' responsiveness, (b) assisted them to maintain a holistic understanding, (c) enabled them to react intuitively, (d) focused their attention, (e) assisted them to monitor their progress, and (f) helped them shift to another approach by attending to another aspect of the image.

Imagery was supported by conceptualisation which, in turn, was aided by comments from the teacher or from other students, and by observations of other students' work. As a result, students manipulated the materials in certain ways. After observing the results and making decisions about the suitability of certain actions, further manipulations were made. Other interactions with materials such as matching pieces or parts of pieces to compare the size of angles assisted the development of the concept of angle and of size of angles. Students were able to test their concepts against the materials and in this way they were able to assess their own progress. They frequently turned the fore-finger away from the thumb in order to consider angle size.

Conclusion

This study has shown that learning about angles is enhanced through problem solving situations that encourage students to focus on angle size. Students check their developing concepts by interacting with peers or the teacher or by manipulating materials. Students are being responsive and engaging in learning, noticing and attending. Some intention is gained from the words of others. Students analyse and note size as part of their early recognition of angles although it is not initially a stable concept. Further focusing, checking, and experiencing of different orientations and contexts assists the concepts to develop. Without responsiveness, the students would not be acting and constructing meaning. The meaning of angle is embedded in numerous conceptual frameworks which develop concurrently and integrally with visual imagery.

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